Construction of the solution of the inverse spectral problem for a system depending rationally on the spectral parameter, Borg-Marchenko-type theorem, and sine-Gordon equation

Alexander Sakhnovich *

Running head: System depending rationally on spectral parameter.

Abstract

Weyl theory for a non-classical system depending rationally on the spectral parameter is treated. Borg-Marchenko-type uniqueness theorem is proved. The solution of the inverse problem is constructed. An application to sine-Gordon equation in laboratory coordinates is given.

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1 Introduction

Canonical systems

$$\frac{d}{dx}w(x,\lambda) = i\lambda JH(x)w(x,\lambda), \quad H \geq 0, \quad J = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \tag{1.1}$$

where H are $2n \times 2n$ matrix functions and I_n is the $n \times n$ identity matrix, are classical objects of analysis, which include Dirac systems, matrix string equations and Schrödinger equations as particular cases. For the literature on canonical systems see, for instance, the books [3, 10, 19, 39] and various references in the papers [22,28–30,34]. We shall consider systems of the form

$$y'(x,\lambda) = i\sum_{k=1}^{m} b_k(\lambda - d_k)^{-1}\Big(\beta_k(x)^*\beta_k(x)\Big)y(x,\lambda), \quad b_k = \pm 1, \quad x \in [0,\infty), \tag{1.2}$$

where $y' = \frac{d}{dx}$, $\beta_k(x) = \begin{bmatrix} \beta_{k1}(x) & \beta_{k2}(x) \end{bmatrix}$ are $\mathbb{C}^2$-valued differentiable vector functions such that

$$\sup_{0<x<\infty} \|\beta_k'(x)\| < \infty, \quad \beta_k(x)\beta_k(x)^* \equiv 1, \quad 1 \leq k \leq m, \tag{1.3}$$

and $\mathbb{C}$ is the complex plane. We shall treat also a somewhat wider class of systems (1.2), such that the vector functions β_k satisfy relations

$$\sup_{0<x<l} \|\beta_k'(x)\| < \infty \text{ for all } 0 < l < \infty, \quad \beta_k(x)\beta_k(x)^* \equiv 1, \quad 1 \leq k \leq m. \tag{1.4}$$

Systems (1.2) generalize a subclass of canonical systems for the important case of several poles d_p with respect to the spectral parameter λ. See, for instance, interesting papers [11, 45] on systems with rational dependence on λ. A system of the form (1.2), where $m = 2$, can be treated as an auxiliary system for the sine-Gordon equation in laboratory coordinates (see Introduction in [26] and Section 6 here). We always assume that

$$d_k = \overline{d}_k \neq d_p \quad \text{for} \quad k \neq p, \quad 1 \leq k, p \leq m, \tag{1.5}$$

where $\overline{d}_k$ is complex conjugate to d_k.

The 2×2 matrix function $w(x, \lambda)$ satisfying (1.2) and the normalization condition, that is,

$$w'(x, \lambda) = i \sum_{k=1}^{m} b_k (\lambda - d_k)^{-1} \beta_k(x)^* \beta_k(x) w(x, \lambda), \quad w(0, \lambda) = I_2 \tag{1.6}$$

is called the fundamental solution of (1.2). Different generalizations of the notion of a Weyl function are based on the asymptotics of the fundamental solution (see e.g. [4, 5, 9, 16, 23–25, 31, 33, 43, 45]).

A k–th Weyl-Titchmarsh function $\phi_k(\lambda)$ of the system (1.2) was introduced in [26] on the complex disk

$$D_M := \left\{ \lambda \in \mathbb{C} : \left| \lambda - d_k - i\frac{b_k}{M} \right| < \frac{1}{M} \right\}.$$

Here, it is more convinient to change variables and use the functions $\varphi_k(\mu) = \phi_k\big(\lambda(\mu)\big)$, where

$$\lambda = d_k + \frac{b_k}{2\mu}, \quad \mu = \frac{b_k}{2(\lambda - d_k)}. \tag{1.7}$$

In view of (1.7) the inequality $\Im\mu < -M/4$ is equivalent to the relation $\lambda \in D_M$.

Definition 1.1 *A function $\varphi_k(\mu)$ is called a WT_k function of the system (1.2) with the properties (1.4) if and only if there exists an $M = M_k > 0$ such that φ_k is holomorphic on the half-plane $\Im\mu < -M/4$ and for all $x \in [0, \infty)$*

$$\sup_{\Im\mu < -M/4} \left\| w\big(x, \lambda(\mu)\big) \begin{bmatrix} \varphi_k(\mu) \\ 1 \end{bmatrix} \right\| < \infty. \tag{1.8}$$

We shall use the notation $\zeta = \Re\mu$, $\eta = \Im\mu$ $(\mu = \zeta + i\eta)$. Here $\Re$ is the real part, $\Im$ is the imaginary part, and the real axis will be denoted by $\mathbb{R}$. When the conditions (1.3) hold, the analyticity of φ_k follows automatically from (1.8).

The direct Weyl-Titchmarsh theory for $m = 2$ (two poles) was treated and the uniqueness of the solution of the inverse problem was proved in [26]. Here we shall construct this unique solution of the inverse spectral problem ($m \geq 2$). Starting from the seminal work [21] by M. Krein structured operators were successfully used to solve inverse spectral problems. In the cases of Krein or self-adjoint Dirac-type systems these were operators with difference kernels [2, 12, 21, 34, 37, 39]. A somewhat more complicated structured operators will appear in this article.

An important series of papers by F. Gesztesy, B. Simon and coauthors on the high energy asymptotics of the Weyl functions and local Borg-Marchenko-type uniqueness results has initiated a growing interest in this important domain (see [7, 8, 14, 15, 17, 34, 41, 42] and references therein). The Weyl-Titchmarsh theory for a non-self-adjoint case (the skew-self-adjoint Dirac type system) has been studied in [6, 18, 31] and the Borg-Marchenko-type results for this system have been published in [35]. Using the procedure to construct the solution of the inverse problem, we obtain here a Borg-Marchenko-type theorem for another interesting non-self-adjoint case, namely, for system (1.2).

Finally, an application to the boundary value problem for sine-Gordon equation in laboratory coordinates (the case of bounded solution) will be given.

Some preliminary results on the existence and uniqueness of the WT_k functions, representation of the fundamental solutions, and uniqueness of solution of the inverse problem are given in the next Section 2. The structured operators, which are necessary to construct the solution of the inverse problem, are studied in Section 3. The construction of the solution of the inverse problem and Borg-Marchenko-type theorem are contained in Section 4. The notion of the Weyl set is introduced in Section 5, and the sine-Gordon equation is treated in Section 6.

2 Preliminaries

To make our paper self-contained we shall formulate in this section some results from [26]. We formulate them for $m \geq 2$ as the proofs for the case $m \geq 2$ are similar to the proofs for the case $m = 2$ treated in [26]. The existence and uniqueness of the Weyl functions are stated in Theorem 2.2 [26].

Theorem 2.1 *Let (1.2) be a system with coefficients β_k which are absolutely continuous vector functions satisfying (1.3) and the additional condition*

$$\beta_{k1}(0) \neq 0 \quad (1 \leq k \leq m). \tag{2.1}$$

Then there exist unique WT_k–functions φ_k $(1 \leq k \leq m)$ of the system (1.2).

To connect WT_k-functions with the solutions from L^2, similar to the classical Weyl functions, let w be the fundamental solution of system (1.2). For fixed k (which we sometimes omit in the notations), we define

$$W(x, \mu) = W_k(x, \mu) := e^{-ix\mu} Q(x) w\big(x, \lambda(\mu)\big), \tag{2.2}$$

where μ and λ are connected by the formula (1.7) and Q is the 2×2 matrix function given by

$$Q(x) = Q_k(x) := \begin{bmatrix} \beta_{k1}(x) & \beta_{k2}(x) \\ -\overline{\beta_{k2}(x)} & \overline{\beta_{k1}(x)} \end{bmatrix}, \qquad x \in [0, \infty). \tag{2.3}$$

Here β_{kj} denote the entries of β_k, that is, $\beta_k = [\beta_{k1} \quad \beta_{k2}]$. By (1.3) and (2.3) the function $Q(x)$ is unitary-valued, that is,

$$Q(x)^* Q(x) = Q(x) Q(x)^* = I_2, \qquad x \in [0, \infty). \tag{2.4}$$

From the second relation in (1.3) and formulas (1.7) and (2.3) it follows that

$$i \frac{b_k}{\lambda - d_k} Q_k(x) \beta_k(x)^* \beta_k(x) Q_k(x)^* = 2i\mu \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda = \lambda(\mu). \tag{2.5}$$

By (1.6), (2.2), and (2.5) the matrix function W satisfies the system

$$W'(x,\mu) = (i\mu j + \xi(x,\mu))\, W(x,\mu), \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \in [0,\infty), \quad (2.6)$$

where

$$\xi(x,\mu) = Q_k'(x)Q_k(x)^* + iQ_k(x)\left(\sum_{p\neq k} \frac{b_p\beta_p(x)^*\beta_p(x)}{\lambda - d_p}\right) Q_k(x)^*. \qquad (2.7)$$

From (2.2), (2.4), (2.6), and (2.7) it follows that

$$W(x,\overline{\mu})^* W(x,\mu) = Q(0)^*Q(0) = I_2, \quad W(0,\mu) = Q(0). \qquad (2.8)$$

The analog of Theorem 2.4 [26], which is formulated below, states that $W_k(x,\mu) \begin{bmatrix} \varphi_k(\mu) \\ 1 \end{bmatrix} \in L_2^2$.

Theorem 2.2 *Let (1.2) be a system with coefficients β_k which are absolutely continuous vector functions satisfying (1.3) and (2.1). Then the WT_k-functions φ_k are unique functions such that for some $M_k > 0$ and all μ satisfying inequality $\Im\mu < -M_k$ we have*

$$\int_0^\infty [\overline{\varphi}_k(\mu) \quad 1] W_k(x,\mu)^* W_k(x,\mu) \begin{bmatrix} \varphi_k(\mu) \\ 1 \end{bmatrix} dx < \infty. \qquad (2.9)$$

We shall need some details from the proof of Theorem 2.2 [26] (Theorem 2.1 here) in our further considerations. Notice that the Dirac-type system can be written down in the form (2.6), where ξ does not depend on μ. Similar to the Dirac-type system case [31], choose a value $M > 0$ such that

$$\sup_{x\in[0,\infty),\Im\mu<-M/4} \|\xi(x,\mu)\| < \frac{1}{4}M. \qquad (2.10)$$

By (2.6) and (2.10) one can see that for $\Im\mu < -M/4$ we have [26]:

$$\frac{d}{dx}R(x,\mu) > 0, \quad R(x,\mu) := Q(0)W(x,\mu)^* jW(x,\mu)Q(0)^*. \qquad (2.11)$$

Now, put

$$\mathfrak{A}(x,\mu) = \mathfrak{A}_k(x,\mu) = \{\mathfrak{A}_{jp}(x,\mu)\}_{j,p=1}^2 := Q(0)W_k(x,\mu)^{-1}. \tag{2.12}$$

According to the second relation in (2.8) and to (2.11) we have $R(x,\mu) > j$ or, equivalently, $\mathfrak{A}(x,\mu)^* j \mathfrak{A}(x,\mu) < j$. Thus, the linear fractional transformation

$$\psi_k(l,\mu) = \frac{\mathfrak{A}_{11}(l,\mu)\theta(\mu) + \mathfrak{A}_{12}(l,\mu)}{\mathfrak{A}_{21}(l,\mu)\theta(\mu) + \mathfrak{A}_{22}(l,\mu)}, \quad |\theta(\mu)| \le 1, \quad l > 0, \quad \Im\mu < -\frac{M}{4}, \tag{2.13}$$

where θ is a holomorphic parameter function, is well-defined, and

$$|\psi_k(l,\mu)| < 1. \tag{2.14}$$

The class of functions $\psi_k(l,\cdot)$ given by (2.13) is denoted by $\mathcal{N}_k(l)$. Using (2.11), it is shown in [26] that $\mathcal{N}_k(l_1) \subset \mathcal{N}_k(l_2)$ for $l_1 > l_2$. For each $l > 0$ and for each μ ($\Im\mu < -\frac{M}{4}$) the values of $\psi_k(l,\mu)$ ($\psi_k \in \mathcal{N}_k(l)$) can be parametrized

$$\psi_k(l,\mu) = \rho_1(l,\mu)^{-1/2}\Theta(l,\mu)\rho_2(l,\mu)^{-1/2} + \rho_0(l,\mu), \qquad l \in (0,\infty), \tag{2.15}$$

$$\rho_0 = -R_{11}^{-1}R_{12}, \quad \rho_1 = R_{11}, \quad \rho_2 = \left(R_{21}R_{11}^{-1}R_{12} - R_{22}\right)^{-1}, \tag{2.16}$$

where $|\Theta(l,\mu)| \le 1$ and R_{jp} are the entries of $R = \{R_{jp}\}_{j,p=1}^2$. The set of values of $\psi_k(l,\mu)$ ($\psi_k \in \mathcal{N}_k(l)$) coincides with the disk on the right-hand side of (2.15), that is, the values of ψ_k form the so called Weyl disks. The functions $\rho_1(l)^{-1/2}$ and $\rho_2(l)^{-1/2}$ are decreasing, and for ρ_1 we have

$$\rho_1(l) \ge 1 - 2l\Big(\frac{M}{4} + \Im\mu\Big) \to \infty, \quad \text{when} \quad l \to \infty. \tag{2.17}$$

Therefore, the intersection of the Weyl disks in (2.15) is a Weyl point, that is, there is only one function $\widetilde{\psi}_k(\mu)$, which belongs to all $\mathcal{N}_k(l)$:

$$\bigcap_{l<\infty} \mathcal{N}_k(l) =: \widetilde{\psi}_k(\cdot). \tag{2.18}$$

To study the asymtotics of $\widetilde{\psi}_k$ we need the representation of the fundamental solution W from Theorem 2.1 [26]:

Theorem 2.3 *Let $\beta_k(x)$ be absolutely continuous $\mathbb{C}^2$-valued vector functions on the interval $[0,\, l]$ $(0 < l < \infty)$ satisfying relations*

$$\sup_{0<x<l} \|\beta_k'(x)\| < \infty, \quad \beta_k(x)\beta_k(x)^* \equiv 1, \quad 1 \leq k \leq m. \tag{2.19}$$

Then $W(x, \mu)$ $(x \in [0,\, l])$ of the form (2.2) admits a representation

$$\begin{aligned} W_k(x,\mu)Q(0)^* = e^{i\mu x j}\Bigg(D_k(x) + \sum_{p\neq k}\left(\mu - \frac{b_k}{2(d_p - d_k)}\right)^{-1} D_p(x)\Bigg) \\ + \int_{-x}^{x} e^{i\mu u} N_k(x,u)du + \sum_{p\neq k}\left(\mu - \frac{b_k}{2(d_p - d_k)}\right)^{-1} \int_{-x}^{x} e^{i\mu u} N_p(x,u)du \\ + \mathrm{O}(\mu^{-2}), \end{aligned} \tag{2.20}$$

for $\mu = \zeta + i\eta, \eta \neq 0,$ $|\zeta| \to \infty$, where D_s $(1 \leq s \leq m)$ are continuous diagonal matrix functions, $D_k^* = D_k^{-1}$, and

$$\sup_{|u|\leq x\leq l} \left(\sum_{s=1}^{m} \|N_s(x,u)\| \right) < \infty. \tag{2.21}$$

From the representation (2.20), after some calculations one gets

$$\lim_{\Im\mu\to-\infty} \widetilde{\psi}_k(\mu) = 0 \tag{2.22}$$

uniformly with respect to $\Re\mu$ (see formula (2.29) in [26]). As (2.22) holds and $\beta_{k1}(0) \neq 0$, there is a value $\widetilde{M}$ such that

$$|\overline{\beta_{k2}(0)}\widetilde{\psi}_k(\mu) + \beta_{k1}(0)| > \varepsilon > 0, \quad \Im\mu < -\widetilde{M} \leq -\frac{M}{4}. \tag{2.23}$$

The WT_k-function is defined in the domain $\Im\mu < -\widetilde{M}$ by the formula

$$\varphi_k(\mu) = \frac{\overline{\beta_{k1}(0)}\widetilde{\psi}_k(\mu) - \beta_{k2}(0)}{\overline{\beta_{k2}(0)}\widetilde{\psi}_k(\mu) + \beta_{k1}(0)}. \tag{2.24}$$

Finally, using (2.15) the estimate

$$|\psi_k(l,\mu) - \breve{\psi}_k(l,\mu)| \leq 2\exp\Big(\big(i(\overline{\mu} - \mu) + M/2\big)l\Big), \quad \Im\mu < -\frac{M}{4} \tag{2.25}$$

was proved for the arbitrary $\psi_k, \breve{\psi}_k \in \mathcal{N}_k(l)$ (see Lemma 2.3 [26]). This estimate, in its turn, helped to prove that the solution of the inverse problem is unique (Theorem 3.1 [26]):

Theorem 2.4 *For given WT_k-functions φ_k $(1 \leq k \leq m)$ there is at most one system (1.2) satisfying conditions (1.3) and (2.1).*

In this paper we shall construct also the solution of the following inverse problem.

Definition 2.5 *The inverse spectral problem for system (1.2), (1.4) is the problem to recover the system, that is, to recover the matrix function*

$$\beta(x) = \begin{bmatrix} \beta_1(x) \\ \cdots \\ \beta_m(x) \end{bmatrix},$$

such that the relations (1.4) and (2.1) hold and that the given functions φ_k are system's WT_k-functions. We call β the potential of system (1.2).

The corresponding uniqueness theorem was proved in [26] using Theorem 2.3.

Theorem 2.6 *For given functions φ_k $(1 \leq k \leq m)$, which admit asymptotic representations*

$$\varphi_k(\mu) = c_k + O(\mu^{-1}), \quad c_k \in \mathbb{C}, \quad \Im \mu < -\frac{M}{4}, \quad \mu \to \infty, \tag{2.26}$$

there is at most one system (1.2) satisfying conditions (1.4) and (2.1) and such that the functions φ_k are the system's WT_k-functions. Moreover, it follows from (2.26) that

$$c_k = -\beta_{k2}(0)/\beta_{k1}(0). \tag{2.27}$$

3 S-nodes

Interesting developments of the classical results on the inverse problems were obtained using the notion of the S-node (see [37,39] and references therein). The non-self-adjoint systems were also recovered from their Weyl functions using S-nodes [31, 33, 35]. Here, we shall consider S-nodes corresponding to system (1.2). By $\{\mathcal{H}_1,\, \mathcal{H}_2\}$ we denote the class of the linear bounded operators acting from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$, diag means diagonal matrix, and $A_0(l) \in \{L^2_m(0,l),\, L^2_m(0,l)\}$ is an integration operator: $(A_0 f)(x) = \int_0^x f(u)du$, $x \in [0,l]$. As in the expression $A_0 f$ above, sometimes we omit l in our notations. Later we shall also denote by $A_0(l)$ the operator of integration in $L^2(0,l)$. We denote the identity operator by I. It is always clear from the context, where I is acting.

Now, we introduce three bounded operators $A(l)$, $S(l)$, and $\Pi(l)$. Operators $A(l)$ have simple structure and do not depend on the choice of $\beta_k(x)$:

$$Af = A(l)f = Df + iBA_0(l)f, \quad A(l) \in \{L^2_m(0,l),\, L^2_m(0,l)\}, \tag{3.1}$$

$$D = \mathrm{diag}\{d_1, d_2, \ldots, d_m\}, \quad B = \mathrm{diag}\{b_1, b_2, \ldots, b_m\}. \tag{3.2}$$

Operator Π is an operator of multiplication by the $m \times 2$ matrix function $[\Phi_1(x) \quad \Phi_2(x)]$:

$$\Pi(l) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = g_1\Phi_1(x) + g_2\Phi_2(x), \quad \Pi(l) \in \{\mathbb{C}^2,\, L^2_m(0,l)\}, \tag{3.3}$$

where the entries of the absolutely continuous column vector functions Φ_p $(p = 1, 2)$ are denoted by Φ_{kp}, and we require

$$\Phi_{k1}(x) \equiv 1, \quad \Phi'_{k2}(x) \in L^2(0,l), \quad 1 \le k \le m. \tag{3.4}$$

Later we shall recover Φ_2 from the Weyl functions.

Operator $S \in \{L^2_m(0,l),\, L^2_m(0,l)\}$ is chosen so that it satisfies the operator identity

$$AS - SA^* = i\Pi\Pi^*. \tag{3.5}$$

Therefore we say that A, S, and Π form an S-node.

Proposition 3.1 *Let operators A and Π be given by equalities (3.1)-(3.3), and let Π satisfy (3.4). Then the unique bounded operator S, which satisfies (3.5), has the form*

$$Sf = B\widetilde{D}f + \int_0^l s(x,u)f(u)du, \quad s(x,u) = \{s_{kp}(x,u)\}_{k,p=1}^m, \tag{3.6}$$

where

$$\widetilde{D} = I_m + \mathrm{diag}\{|\Phi_{12}(0)|^2, |\Phi_{22}(0)|^2, \dots, |\Phi_{m2}(0)|^2\}. \tag{3.7}$$

The entries of $s(x,u)$ on the main diagonal are defined by the equalities

$$s_{kk}(x,u) = \frac{b_k}{2}\int_{|x-u|}^{x+u} \Phi_{k2}'\left(\frac{v+x-u}{2}\right)\overline{\Phi_{k2}'\left(\frac{v+u-x}{2}\right)}dv. \tag{3.8}$$

The offdiagonal ($k \neq p$) entries of $s(x,u)$ are defined by the equalities

$$s_{kp}(x,u) = \frac{1}{2\pi}\int_\Gamma \Big((\lambda I - A_k)^{-1}[1 \quad \Phi_{k2}]\Big)(x)\Big((\overline{\lambda}I - A_p)^{-1}[1 \quad \Phi_{p2}]\Big)(u)^* d\lambda, \tag{3.9}$$

where $\Gamma = \{\lambda : |\lambda - d_k| = \varepsilon > 0\}$ is anticlockwise oriented, $\varepsilon < |d_k - d_p|$, I is the identity operator, $A_k = d_k I + i b_k A_0$, and we use A_0 here to denote the integration operator in $L^2(0,l)$.

Proof

Write down S in the matrix form

$$S = \{S_{kp}\}_{k,p=1}^m, \qquad S_{kp} \in \{L^2(0,l),\, L^2(0,l)\}.$$

Then identity (3.5) takes the form

$$A_0 S_{kk} + S_{kk}A_0^* = b_k \Pi_k \Pi_k^*, \quad \Pi_k \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = g_1 + g_2\Phi_{k2}(x), \quad \Pi_k \in \{\mathbb{C}^2,\, L^2(0,l)\}, \tag{3.10}$$

$$A_k S_{kp} - S_{kp}A_p^* = i\Pi_k \Pi_p^*, \quad A_k = d_k I + i b_k A_0, \quad k \neq p. \tag{3.11}$$

The bounded solution T of the equation $TA_0 + A_0^*T = Q$ for Q of the form $Q = \int_0^l q(x,t)\cdot dt$ is constructed in Theorem 1.3 (p.11) [38]. After easy

transformations we derive from this result the solution of (3.10) too. Namely, we have

$$S_{kk}f = \frac{b_k}{2}\frac{d}{dx}\int_0^l \frac{\partial}{\partial u}\left(\int_{|x-u|}^{x+u}\Big(1+\Upsilon(v,x,u)\Big)dv\right)f(u)du, \tag{3.12}$$
$$\Upsilon(v,x,u) := \Phi_{k2}\big((v+x-u)/2\big)\overline{\Phi_{k2}\big((v-x+u)/2\big)}.$$

From (3.12), taking into account the second relation in (3.4), we derive (3.6)-(3.8). To get (3.9) rewrite (3.11) in the form

$$(\lambda I - A_k)^{-1}S_{kp} - S_{kp}(\lambda I - A_p^*)^{-1} = i(\lambda I - A_k)^{-1}\Pi_k\Pi_p^*(\lambda I - A_p^*)^{-1}, \tag{3.13}$$

and notice that $\sigma(A_k) = d_k$ and $\sigma(A_p^*) = d_p$, where $d_k \neq d_p$ and σ means spectrum. So, we recover S_{kp} by integration of the both parts of (3.13) in the small neighborhood of d_k:

$$S_{kp} = \frac{1}{2\pi}\int_\Gamma (\lambda I - A_k)^{-1}\Pi_k\Pi_p^*(\lambda I - A_p^*)^{-1}d\lambda. \tag{3.14}$$

In other words we have

$$S_{kp} = \int_0^l s_{kp}(x,u)\cdot du, \tag{3.15}$$

where s_{kp} satisfies (3.9). □

Remark 3.2 *It is easy to check the explicit formula for the resolvent of A_k:*

$$\Big((\lambda I - A_k)^{-1}f\Big)(x) \tag{3.16}$$
$$= (\lambda - d_k)^{-1}f(x) + ib_k(\lambda - d_k)^{-2}\int_0^x \exp\left(\frac{ib_k(x-u)}{\lambda - d_k}\right)f(u)du.$$

Denote by P_r $(r \leq l)$ the orthogonal projector from $L^2_m(0,l)$ onto $L^2_m(0,r)$, that is, let $P_r \in \{L^2_m(0,l),\, L^2_m(0,r)\}$ and let $(P_rf)(x) = f(x)$ for $x \in (0,r)$. Notice that $P_rA(l) = A(r)P_r$. Therefore, we get

$$A(r)P_rSP_r^* - P_rSP_r^*A(r)^* = i\Pi(r)\Pi(r)^*, \tag{3.17}$$

by applying P_r from the left and P_r^* from the right to the both parts of (3.5).

Remark 3.3 *According to (3.17), the unique operator $S(r)$ satisfying the identity*

$$A(r)S(r) - S(r)A(r)^* = i\Pi(r)\Pi(r)^*, \quad r < l, \tag{3.18}$$

is given by the formula

$$S(r) = P_r S P_r^* = B\widetilde{D} + \int_0^r s(x,u) \cdot du, \tag{3.19}$$

where s does not depend on r and $B\widetilde{D}$ means the operator of multiplication by the matrix $B\widetilde{D}$.

We shall need some properties of $S(l)$.

Proposition 3.4 *The operator S constructed in Proposition 3.1 is self–adjoint, boundedly invertible and S^{-1} admits a triangular factorization*

$$S^{-1} = V^* B V, \quad (Vf)(x) = \widetilde{D}^{-\frac{1}{2}} f(x) + \int_0^x V(x,u) f(u) du, \tag{3.20}$$

where

$$V(r,u) = B\widetilde{D}^{\frac{1}{2}} T_r(r,u) \quad (r \geq u), \tag{3.21}$$

and T_r is the matrix kernel of the integral operator

$$T(r) = S(r)^{-1} = B\widetilde{D}^{-1} + \int_0^r T_r(x,u) \cdot du. \tag{3.22}$$

Proof

The operator S is self-adjoint as the unique solution of (3.5). (One could also prove it by (3.6)-(3.9).) The invertibility of S is proved by contradiction. Suppose that S is not invertible. In view of the special structure (3.6) of S, it means that S has an eigenvector $f \neq 0$, such that $Sf = 0$. Taking into account identity (3.5) and equality $Sf = 0$, we derive $\big(f, \Pi\Pi^* f\big)_{L^2} = 0$, where $\big(\cdot, \cdot\big)_{L^2}$ denotes the scalar product in $L^2_m(0,l)$. It is immediate that $\Pi^* f = 0$. Apply the both parts of (3.5) to f and use the equalities $Sf = 0$, $\Pi^* f = 0$ to obtain $SA^* f = 0$. So, from Sf=0 it follows that $SA^* f = 0$.

In other words, we have SL=0 for the linear span L of the vectors $(A^*)^k f$ ($k \geq 0$). Therefore, we have dim $L < \infty$. As $A^*L \subseteq L$ and dim $L < \infty$, there is an eigenvector g of A^*: $A^*g = cg$, $g \neq 0$, and $g \in L$. Hence, by the definition of A_k in (3.11), there is an eigenvector of integration in $L^2(0,l)$: $A_0^* g_k = \widetilde{c} g_k$, $g_k \neq 0$. This is impossible, and so we come to a contradiction, that is, S is invertible.

In view of (3.18) and (3.19) the invertibility of the operators $P_r S P_r^*$ ($r < l$) is proved quite similar to the invertibility of S. By (3.8) and (3.9) the function $s(x,u)$ is continuous. Thus, the factorization conditions from "result 2" Section IV.7 [19] are fulfilled for $\widetilde{S}^{-1}$, where $\widetilde{S} = B\widetilde{D}^{-\frac{1}{2}} S \widetilde{D}^{-\frac{1}{2}}$. Hence, the factorization formula for S^{-1} in (3.20), the second relation in (3.20), and equality (3.21) follow. $\square$

Remark 3.5 *Let the conditions of Proposition 3.1 be fulfilled and put*

$$\beta(x) = \begin{bmatrix} \beta_1(x) \\ \cdots \\ \beta_m(x) \end{bmatrix} = (V\Phi)(x), \quad \Phi(x) := [\Phi_1(x) \quad \Phi_2(x)] \quad (0 \leq x \leq l), \tag{3.23}$$

where V is applied to $\Phi(x)$ columnwise. In other words, we have

$$V\Pi g = \beta(x) g \quad (g \in \mathbb{C}^2). \tag{3.24}$$

Then the matrix functions β_k satisfy the second relation in (1.4), that is, $\beta_k \beta_k^* \equiv 1$. Indeed, from (3.5) and the first equality in (3.20), it follows that

$$V^*BVA - A^*V^*BV = iV^*BV\Pi\Pi^*V^*BV, \quad \text{i.e.,}$$

$$VAV^{-1}B - B(V^*)^{-1}A^*V^* = iV\Pi\Pi^*V^*. \tag{3.25}$$

By the definitions (3.1) and (3.20) of A and V, the operator $VAV^{-1}B$ is lower triangular and has the form $DB + \int_0^x \gamma(x,u) \cdot du$. The operator $B(V^*)^{-1}A^*V^*$ is upper triangular. Hence, one can derive the kernel γ of the integral term of $VAV^{-1}B$ from (3.24) and (3.25). We get

$$(VAV^{-1}f)(x) = Df(x) + i\beta(x) \int_0^x \beta(u)^* B f(u) du. \tag{3.26}$$

By (3.26) it is immediate that

$$VA = DV + i\beta(x)\int_0^x \beta(u)^* BV \cdot du. \tag{3.27}$$

Rewrite (3.27) as the equality of the kernels of the corresponding integral operators:

$$i\widetilde{D}^{-\frac{1}{2}}B + V(x,u)D + i\int_u^x V(x,v)Bdv$$
$$= DV(x,u) + i\beta(x)\Big(\beta(u)^* B\widetilde{D}^{-\frac{1}{2}} + \int_u^x \beta(v)^* BV(v,u)dv\Big). \tag{3.28}$$

When $x = u$, the equality of the main diagonals of the both sides of (3.28) implies $\beta_k\beta_k^* \equiv 1$.

Now, introduce the transfer matrix functions in the Lev Sakhnovich form [36, 37, 39]:

$$w_A(r,\lambda) = I_2 - i\Pi(r)^* S(r)^{-1}\big(A(r) - \lambda I\big)^{-1}\Pi(r). \tag{3.29}$$

The following lemma is essential for the solution of the inverse problem.

Lemma 3.6 *Let the conditions of Proposition 3.1 be fulfilled, and let the S-node be given by the formulas (3.1)-(3.4) and (3.6)-(3.9). Then we have*

$$\frac{d}{dr}w_A(r,\lambda) = i\beta(r)^* B(\lambda I_m - D)^{-1}\beta(r)w_A(r,\lambda). \tag{3.30}$$

Proof

First, introduce several notations. Let $P_1(r,\delta)$ and $P_2(r,\delta)$ denote ortoprojectors from $L_m^2(0, r+\delta)$ on $L_m^2(0,r)$ and $L_m^2(r, r+\delta)$, respectively. That is, let $P_1(r,\delta)f \in L_m^2(0,r)$, $P_2(r,\delta)f \in L_m^2(r, r+\delta)$, and

$$\Big(P_1(r,\delta)f\Big)(x) = f(x),\ 0 < x < r; \quad \Big(P_2(r,\delta)f\Big)(x) = f(x),\ r < x < r+\delta. \tag{3.31}$$

For operators K acting in $L^2_m(0, r+\delta)$ we put $K_{jp} := P_j(r,\delta)KP_p(r,\delta)^*$ $(j,p=1,2)$. In particular, we use notations

$$T(u) := S(u)^{-1}, \quad T_{22} := P_2(r,\delta)T(r+\delta)P_2(r,\delta)^*. \tag{3.32}$$

According to [36] (see also Theorem 2.1 from Chapter 1 in [39]) we have

$$\begin{aligned} w_A(r+\delta,\lambda) \quad - \quad & w_A(r,\lambda) = -i\Pi(r+\delta)^*S(r+\delta)^{-1} \\ & \times(A_{22}-\lambda I)^{-1}T_{22}^{-1}P_2(r,\delta)S(r+\delta)^{-1}\Pi(r+\delta)w_A(r,\lambda). \end{aligned} \tag{3.33}$$

Using formula (3.23) and the first equality in (3.20), we rewrite (3.33) in the form

$$w_A(r+\delta,\lambda) - w_A(r,\lambda) = i\int_r^{r+\delta}\beta(x)^*(Z_\delta\beta)(x)dx\, w_A(r,\lambda), \tag{3.34}$$

where the operator $Z_\delta \in \{L^2_m(r,r+\delta),\, L^2_m(r,r+\delta)\}$ is given by the formula

$$Z_\delta = BV(\lambda I - A_{22})^{-1}T_{22}^{-1}P_2(r,\delta)V^*B. \tag{3.35}$$

It is easy to see that $Z_\delta - B(\lambda I - D)^{-1}$ is an integral operator, and we shall show below that the kernel of this operator is bounded:

$$Z_\delta = B(\lambda I - D)^{-1} + \int_r^{r+\delta} z_\delta(x,u)\cdot du, \quad \sup \|z_\delta(x,u)\| < \infty, \tag{3.36}$$

where $r \le x, u \le r+\delta \le l$. For that purpose notice that the kernel $s(x,u)$ is continuous, and according to Section IV.7 [19] the kernel $T_r(x,u)$ is continuous with respect to r, x, and u $(x,u \le r \le l)$ too. Hence, the functions $V(x,u)$ and $\beta(x)$ are continuous, and we also have

$$\sup_{x,u\le l}\|s(x,u)\| < \infty, \qquad \sup_{x,u\le r\le l}\|T_r(x,u)\| < \infty. \tag{3.37}$$

From the definition (3.32) we get

$$T_{22}^{-1} = S_{22} - S_{21}S_{11}^{-1}S_{12}, \tag{3.38}$$

and so, by (3.37) the kernel of the integral term of T_{22}^{-1} is bounded. In view of (3.16), for the k-th entry of $(\lambda I - A_{22})^{-1}f$ we have

$$\begin{aligned} &\Big((\lambda I - A_{22})^{-1}f\Big)_k(x) \\ &\quad = (\lambda - d_k)^{-1}f(x) + ib_k(\lambda-d_k)^{-2}\int_r^x \exp\left(\frac{ib_k(x-u)}{\lambda - d_k}\right)f(u)du, \end{aligned} \tag{3.39}$$

and the kernel of $(\lambda I - A_{22})^{-1} - (\lambda I - D)^{-1}$ is bounded for any fixed λ $(\lambda \neq d_k)$ too. Therefore, for Z_δ given by (3.35) the formula (3.36) is true. Recall that β is continuous. Thus, from (3.34) and (3.36) it follows that

$$\lim_{\delta \to +0} \delta^{-1}\Big(w_A(r+\delta, \lambda) - w_A(r, \lambda)\Big) = i\beta(r)^* B(\lambda I_m - D)^{-1}\beta(r) w_A(r, \lambda). \quad (3.40)$$

Quite similar one can prove that

$$\lim_{\delta \to +0} \delta^{-1}\Big(w_A(r, \lambda) - w_A(r-\delta, \lambda)\Big) = i\beta(r)^* B(\lambda I_m - D)^{-1}\beta(r) w_A(r, \lambda). \quad (3.41)$$

By (3.40) and (3.41) equality (3.30) holds. □

Remark 3.7 *It is easy to see that definition (3.29) implies* $\lim_{r \to 0} w_A(r, \lambda) = I_2$. *Hence, the matrix function* w_A*, which is treated in Lemma 3.6, is the fundamental solution of the system (1.2) corresponding to* $\beta(x) = (V\Phi)(x)$.

4 Inverse problems: construction of the solution

According to formulas (2.14), (2.23), and (2.24), one can choose a sufficiently large value $M > 0$, so that all the WT_k-functions $(1 \leq k \leq m)$ are well-defined and bounded in the half-plane $\Im\mu < -M/4$.

Definition 4.1 *The bounded matrix function*

$$\varphi(\mu) = \mathrm{col}[\varphi_1(\mu), \quad \varphi_2(\mu), \quad \ldots, \quad \varphi_m(\mu)], \quad (4.1)$$

$$\sup_{\Im\mu < -M/4} \|\varphi(\mu)\| < \infty, \quad (4.2)$$

where col means column, is called the Weyl function of system (1.2).

One of our main results is the next theorem.

Theorem 4.2 *System (1.2) satisfying conditions (1.3) and (2.1) is uniquely recovered from its Weyl function. To recover (1.2) on an arbitrary fixed interval* $[0, l]$ *we use the following procedure.*

First, introduce the column vector function $\Phi_2(x)$ $(0 < x < \infty)$ *by the Fourier transform:*

$$\Phi_2(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \mu^{-1} e^{i\mu x} \varphi\left(\frac{\mu}{2}\right) d\zeta \quad (\mu = \zeta + i\eta, \quad \eta < -\frac{M}{2}). \tag{4.3}$$

That is, we define Φ_2 *on* $(0, l)$ *and* $(0, \infty)$ *via norm limit l.i.m. in* L_2^m*:*

$$\Phi_2(x) = \frac{i}{2\pi} e^{-\eta x} \mathrm{l.i.m.}_{a\to\infty} \int_{-a}^{a} \mu^{-1} e^{i\zeta x} \varphi\left(\frac{\mu}{2}\right) d\zeta. \tag{4.4}$$

Here the right-hand side of (4.4) equals 0 *for* $x < 0$*.*

Next, substitute $\Phi_2(x)$ *into formulas (3.7)-(3.9) to introduce operator* $S \in \{L_m^2(0,l),\, L_m^2(0,l)\}$ *of the form (3.6). Apply formulas (3.20)-(3.22) to recover operator* V *from the operators* $S(r) = P_r S P_r^*$ $(r \leq l)$*. Then, the matrix functions* $\beta_k(x)$ *and therefore, system (1.2) is recovered by the formula (3.23), that is,* $\beta(x) = V[\Phi_1(x) \quad \Phi_2(x)]$*, where* Φ_1 *is given by the first relation in (3.4).*

Proof

Step 1. Let system (1.2) satisfy conditions (1.3) and (2.1) and let φ be the Weyl function of system (1.2). According to the inequality (4.2), the vector function Φ_2 is well-defined by (4.3) and does not depend on the choice of $\eta < -\frac{M}{4}$. So, we can fix some $\eta < -\frac{M}{4}$. To show that Φ_2 is absolutely continuous, introduce functions

$$\widehat{\psi}_k(\mu) := \frac{\mathfrak{A}_{12}(l,\mu)}{\mathfrak{A}_{22}(l,\mu)} \in \mathcal{N}_k(l), \quad \widehat{\varphi}_k(\mu) = \frac{\overline{\beta_{k1}(0)}\widehat{\psi}_k(\mu) - \beta_{k2}(0)}{\overline{\beta_{k2}(0)}\widehat{\psi}_k(\mu) + \beta_{k1}(0)}. \tag{4.5}$$

By (2.8) and (2.12) we have $\mathfrak{A}(l,\mu) = Q_k(0) W_k(l,\overline{\mu})^*$. Hence, by Theorem 2.3 the matrix function $\widehat{\psi}_k$ admits representation

$$\widehat{\psi}_k(\mu) = \left(\overline{d_{k22}(l)}\right)^{-1} \int_{-l}^{l} e^{-i\mu(l+u)} \overline{N_{k21}(l,u)} du + g_1(\mu), \tag{4.6}$$

where d_{k22} is the corresponding entry of the diagonal unitary matrix D_k, N_{k21} is an entry of the bounded matrix function N_k, and the function $g_1(\zeta+i\eta) \in L^1(-\infty,\,\infty)\cap L^\infty(-\infty,\,\infty)$ with respect to the variable ζ. (Recall that L^∞ is the space of bounded functions.) Taking into account (2.23) and (2.25), without loss of generality we can assume

$$|\overline{\beta_{k2}(0)}\widehat{\psi}_k(\mu)+\beta_{k1}(0)| > \varepsilon > 0, \quad \Im\mu < -\frac{M}{4}. \tag{4.7}$$

In view of relations (4.5)-(4.7) we get

$$\widehat{\varphi}_k(\mu) = -\frac{\beta_{k2}(0)}{\beta_{k1}(0)} + \frac{|\beta_{k1}(0)|^2-|\beta_{k2}(0)|^2}{\beta_{k1}(0)^2}\widehat{\psi}_k(\mu)+g_2(\mu), \tag{4.8}$$

where $g_2(\zeta+i\eta) \in L^1(-\infty,\,\infty)\cap L^\infty(-\infty,\,\infty)$. Finally, notice that uniformly for x on the intervals $[\delta,\,l]$ we have

$$\lim_{a\to\infty}\frac{i}{2\pi}\int_{-a}^{a}\mu^{-1}e^{i\mu x}d\zeta = -1 \quad (\mu=\zeta+i\eta, \quad \eta<-\frac{M}{2}). \tag{4.9}$$

From (4.6), (4.8), and (4.9) it follows that the function

$$\widehat{\Phi}_{k2}(x) := \frac{i}{2\pi}\int_{-\infty}^{\infty}\mu^{-1}e^{i\mu x}\widehat{\varphi}_k\left(\frac{\mu}{2}\right)d\zeta \quad (\mu=\zeta+i\eta, \quad \eta<-\frac{M}{2}), \tag{4.10}$$

is absolutely continuous and

$$\sup_{0<x<l}|\widehat{\Phi}_{k2}'(x)| < \infty. \tag{4.11}$$

Next, we shall show that

$$\widehat{\Phi}_{k2}(x) = \Phi_{k2}(x) \quad \text{for} \quad 0 \le x \le l, \tag{4.12}$$

where Φ_{k2} denotes the k-th entry of the $\mathbb{C}^m$-valued vector function Φ_2. The proof of (4.12) requires some considerations. According to the definitions (2.24) and (4.5) of φ_k and $\widehat{\varphi}_k$, respectively, we have

$$\varphi_k(\mu)-\widehat{\varphi}_k(\mu) = \frac{\big(|\beta_{k1}(0)|^2+|\beta_{k2}(0)|^2\big)\big(\widetilde{\psi}_k(\mu)-\widehat{\psi}_k(\mu)\big)}{\big(\overline{\beta_{k2}(0)}\widetilde{\psi}_k(\mu)+\beta_{k1}(0)\big)\big(\overline{\beta_{k2}(0)}\widehat{\psi}_k(\mu)+\beta_{k1}(0)\big)}. \tag{4.13}$$

In view of (2.23), (2.25), and (4.7), formula (4.13) implies for some $C, \widetilde{M} > 0$ that

$$|\varphi_k(\mu) - \widehat{\varphi}_k(\mu)| < C(l)e^{2\eta l}, \quad \eta < -\widetilde{M}(l) < -\frac{M}{4} \quad (\mu = \zeta + i\eta). \qquad (4.14)$$

Taking into account (4.2) and (4.14), one can see that functions $\mu^{-1}\varphi_k\left(\frac{\mu}{2}\right)$ and $\mu^{-1}\widehat{\varphi}_k\left(\frac{\mu}{2}\right)$ are holomorphic in the half-plane $\eta = \Im\mu < -2\widetilde{M}(l)$ and belong to $L^2(-\infty, \infty)$ with respect to $\zeta = \Re\mu$ for any fixed $\eta < -2\widetilde{M}(l)$. Therefore, according to Theorem V [27] these functions admit Fourier representations

$$\mu^{-1}\varphi_k\left(\frac{\mu}{2}\right) = \mathrm{l.i.m.}_{a\to\infty}\int_0^a e^{-i\mu x}e^{2x\widetilde{M}}f(x)dx, \quad f \in L^2(0,\infty), \qquad (4.15)$$

$$\mu^{-1}\widehat{\varphi}_k\left(\frac{\mu}{2}\right) = \mathrm{l.i.m.}_{a\to\infty}\int_0^a e^{-i\mu x}e^{2x\widetilde{M}}\widehat{f}(x)dx, \quad \widehat{f} \in L^2(0,\infty). \qquad (4.16)$$

Using Plansherel's theorem and formulas (4.3) and(4.10), we express $f(x)$ and $\widehat{f}(x)$ in (4.15) and (4.16) via $\Phi_{k2}(x)$ and $\widehat{\Phi}_{k2}(x)$, respectively, and obtain:

$$\frac{i}{\mu}\varphi_k\left(\frac{\mu}{2}\right) = \int_0^\infty e^{-i\mu x}\Phi_{k2}(x)dx, \quad e^{-2x\widetilde{M}}\Phi_{k2}(x) \in L^2(0,\infty), \qquad (4.17)$$

$$\frac{i}{\mu}\widehat{\varphi}_k\left(\frac{\mu}{2}\right) = \int_0^\infty e^{-i\mu x}\widehat{\Phi}_{k2}(x)dx, \quad e^{-2x\widetilde{M}}\widehat{\Phi}_{k2}(x) \in L^2(0,\infty). \qquad (4.18)$$

Consider now the entire matrix function

$$Y_0(\mu) = e^{i\mu l}\int_0^l e^{-i\mu x}\Big(\Phi_{k2}(x) - \widehat{\Phi}_{k2}(x)\Big)dx = \int_0^l e^{i\mu(l-x)}\Big(\Phi_{k2}(x) - \widehat{\Phi}_{k2}(x)\Big)dx. \qquad (4.19)$$

From (4.17)-(4.19) it follows that for $\mu = \zeta + i\eta$, $\eta < -2\widetilde{M}$ we have

$$|Y_0(\mu)| \leq \int_l^\infty e^{\eta(x-l))}\Big(|\Phi_{k2}(x)| + |\widehat{\Phi}_{k2}(x)|\Big)dx + e^{-\eta l}\left|\frac{i}{\mu}\Big(\varphi_k\left(\frac{\mu}{2}\right) - \widehat{\varphi}_k\left(\frac{\mu}{2}\right)\Big)\right|. \qquad (4.20)$$

Recall that Φ_{k2}, $\widehat{\Phi}_{k2} \in L^2(0,\infty)$. Hence, from (4.14) and (4.20) we derive

$$\sup_{\eta<-2\widetilde{M}-\varepsilon}|Y_0(\mu)| < \infty \quad (\varepsilon > 0), \qquad \lim_{\eta\to-\infty}|Y_0(\mu)| = 0 \quad (\mu = \zeta + i\eta). \qquad (4.21)$$

By the definition (4.19) and by the first relation in (4.21), the entire functon Y_0 is bounded in $\mathbb{C}$. So, in view of the second relation in (4.21) we have $Y_0 = 0$, and equality (4.12) is immediate.

From (4.11) and (4.12) we get the second relation in (3.4). Therefore, Π satisfies the conditions of Proposition 3.1, and the S-node and the operator V in our theorem are well-defined. As the conditions of Propositions 3.1 and 3.4 are satisfied, the matrix function $\breve{\beta}(x) = (V\Phi)(x)$ is well-defined and unique. To prove the theorem means to prove the equalities

$$\breve{\beta}_k(x)^*\breve{\beta}_k(x) \equiv \beta_k(x)^*\beta_k(x) \quad (1 \le k \le m). \tag{4.22}$$

Step 2. By (4.10), taking into account (4.6), (4.8), and (4.9), we have

$$e^{\eta x}\widehat{\Phi}_{k2}'(x) = -\frac{1}{2\pi}\mathrm{l.i.m.}_{a\to\infty}\int_{-a}^{a} e^{i\zeta x}\left(\widehat{\varphi}_k\Big(\frac{\mu}{2}\Big) + \frac{\beta_{k2}(0)}{\beta_{k1}(0)}\right)d\zeta, \tag{4.23}$$

$$e^{\eta x}\widehat{\Phi}_{k2}'(x) \in L^2(0,\infty)\cap L^\infty(0,\infty), \quad \eta < -\frac{M}{2}. \tag{4.24}$$

Introduce a 2×2 matrix function:

$$\Omega(\mu) = [\Omega_1(\mu) \quad \Omega_2(\mu)] = Q(0)^*\left[\begin{array}{cc} \exp\{-2ir\mu\} & \widehat{\psi}_k(\mu) \\ 0 & 1 \end{array}\right]. \tag{4.25}$$

First, let us show that for $r \le l$ and for sufficiently large M the inequalities

$$\sup_{\Im\mu < -M/4}\|w_A\big(r,\lambda(\mu)\big)\Omega_1(\mu)\| < \infty, \qquad \sup_{\Im\mu<-M/4}\|w_A\big(r,\lambda(\mu)\big)\Omega_2(\mu)\| < \infty \tag{4.26}$$

are true. Indeed, by Lemma 3.6 w_A satisfies system (1.2), where we substitute $\breve{\beta}_k$ instead of β_k. Hence, for sufficiently large M we get

$$\begin{aligned} &\frac{d}{dr}\Big((\exp\{-2ir(\mu-\overline{\mu})\})w_A(r,\lambda)^*w_A(r,\lambda)\Big) \qquad (4.27)\\ &= \exp\{-2ir(\mu-\overline{\mu})\}w_A(r,\lambda)^*\Big(4\eta\big(I_2 - \breve{\beta}_k(r)^*\breve{\beta}_k(r)\big) + g_3(r,\lambda)\Big)w_A(r,\lambda),\\ &\lambda = \lambda(\mu), \quad g_3(r,\lambda) < CI_2 \quad (\eta = \Im\mu < -M/4), \end{aligned}$$

for some $C > 0$, which does not depend on r. As $\eta(I_2 - \breve{\beta}_k(r)^*\breve{\beta}_k(r)) \leq 0$, formula (4.27) implies

$$\sup_{\Im\mu < -M/4} \| \exp\{-2ir\mu\} w_A(r, \lambda)\| < \infty, \tag{4.28}$$

that is, the first inequality in (4.26) holds. Next, let us prove that

$$\sup_{\Im\mu < -M/4} \|w_A(r, \lambda) \begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix} \| < \infty \quad (r \leq l). \tag{4.29}$$

For that purpose consider w_A again and notice that from the operator identity (3.18) and definition (3.29) it follows [37] that

$$\begin{aligned} &w_A(r, \lambda)^* w_A(r, \lambda) \\ &= I_2 - i(\lambda - \overline{\lambda})\Pi(r)^*\big(A(r)^* - \overline{\lambda}I\big)^{-1} S(r)^{-1}\big(A(r) - \lambda I\big)^{-1}\Pi(r). \end{aligned} \tag{4.30}$$

By (3.4) and (3.16) we derive the equality:

$$\big(A_k(r) - \lambda I\big)^{-1}[\Phi_{k1} \quad \Phi_{k2}] = -2b_k\mu e^{2i\mu x}\Big[1 \quad \Phi_{k2}(0) + \int_0^x e^{-2i\mu u}\Phi_{k2}'(u)du\Big], \tag{4.31}$$

which helps to estimate the right-hand side of (4.30). According to (4.18) and (4.24) the equality

$$\widehat{\varphi}_k\,(\mu) = -\widehat{\Phi}_{k2}(0) - \int_0^\infty e^{-2i\mu u}\widehat{\Phi}_{k2}'(u)du \tag{4.32}$$

is true. In view of (4.12), (4.31), and (4.32) we have

$$\big(A_k(r) - \lambda I\big)^{-1}\Pi_k \begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix} = 2b_k\mu e^{2i\mu x}\int_x^\infty e^{-2i\mu u}\widehat{\Phi}_{k2}'(u)du. \tag{4.33}$$

By (4.24) and (4.33) for sufficiently large M we get

$$\left\|\big(A_k(r) - \lambda I\big)^{-1}\Pi_k \begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix}\right\| \leq C(r)|\mu/\eta|. \tag{4.34}$$

Moreover, for sufficiently large M the functions $\widehat{\varphi}_k(\mu)$ and resolvents $\big(A_p(r) - \lambda I\big)^{-1}$ $(p \neq k)$ are uniformly bounded in the domain $\Im\mu < -M/4$. Therefore, the inequalities (4.34) imply

$$\left\|\big(A(r) - \lambda I\big)^{-1}\Pi \begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix}\right\| \leq C_1(r)|\mu/\eta|. \tag{4.35}$$

Notice that for $\lambda = d_k + \frac{b_k}{2\mu}$ we have $|\lambda - \overline{\lambda}| = -\eta|\mu|^{-2}$. Hence, from (4.30) and (4.35) follows (4.29).

By (4.5) we have

$$\begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix} = Q(0)^* \begin{bmatrix} \widehat{\psi}_k(\mu) \\ 1 \end{bmatrix} \big(\overline{\beta_{k2}(0)}\widehat{\psi}_k(\mu) + \beta_{k1}(0)\big)^{-1}. \tag{4.36}$$

Substitute (4.36) into (4.29) and use (2.14) to derive the second relation in (4.26). Thus, (4.26) is valid.

Now, let us show that

$$\sup_{\Im\mu < -M/4} \|w\big(r, \lambda(\mu)\big)\Omega(\mu)\| < \infty. \tag{4.37}$$

Here, the inequality

$$\sup_{\Im\mu < -M/4} \|\exp\{-2ir\mu\}w(r,\lambda)\| < \infty, \quad \lambda = \lambda(\mu) \tag{4.38}$$

is proved similar to (4.28).

As φ of the form (4.1) is the Weyl function, so φ_k satisfies (1.8). The inequality

$$\sup_{\Im\mu < -M/4} \|w(r,\lambda) \begin{bmatrix} \widehat{\varphi}_k(\mu) \\ 1 \end{bmatrix} \| < \infty \quad (r \le l) \tag{4.39}$$

follows for sufficiently large M from (1.8), (4.14), and (4.38). In view of (2.14) and (4.36), formula (4.39) yields

$$\sup_{\Im\mu < -M/4} \|w(r,\lambda)\Omega_2(\mu)\| < \infty. \tag{4.40}$$

By (4.38) and (4.40) inequality (4.37) is valid.

Step 3. Here, using (4.26) and (4.37) we shall show that

$$\sup_{|\Im\mu| > M/4} \|w_A(r,\lambda)w(r,\lambda)^{-1}\| < \infty. \tag{4.41}$$

First, taking into account (2.2) and (2.20), we obtain

$$w(r,\lambda)Q(0)^* = e^{ir\mu}Q(r)^*W(r,\mu)Q(0)^* = Q(r)^*D_k(r)\mathrm{diag}\{e^{2ir\mu},1\} + g_4(\mu), \tag{4.42}$$

where the 2×2 matrix function $g_4(\zeta+i\eta)$ belongs $L^2_{2\times 2}(-\infty,\infty)$ for each fixed $\eta < -M/4$, that is, the entries of g_4 belong L^2. According to (2.14) and (4.6) the function $\widehat{\psi}(\zeta+i\eta)$, where $\eta<-M/4$, is bounded and belongs to $L^2(-\infty,\infty)$ with respect to ζ. Therefore, formulas (4.25) and (4.42) imply that for $\eta<-M/4$ we have

$$w(r,\lambda)\Omega(\mu) = Q(r)^*D_k(r) + g_5(\mu), \quad g_5(\zeta+i\eta)\in L^2_{2\times 2}(-\infty,\infty). \tag{4.43}$$

After some evident change of variables in Theorem VIII from [27], by (4.37) and (4.43) we can apply it to $w(r,\lambda)\Omega(\mu) - Q(r)^*D_k(r)$. That is, we derive for $\eta<-M/4$ the representation

$$w(r,\lambda)\Omega(\mu) = Q(r)^*D_k(r) + \int_0^\infty e^{-ix\mu}f(x)dx, \quad e^{\eta x}f(x)\in L^2_{2\times 2}(-\infty,\infty). \tag{4.44}$$

In view of (4.26) and (4.44) for sufficiently large M we have

$$\sup_{\Im\mu<-M/4}\|w_A(r,\lambda)w(r,\lambda)^{-1}\| < \infty. \tag{4.45}$$

According to (1.6) and Lemma 3.6 the equalities

$$w(r,\overline{\lambda})^* = w(r,\lambda)^{-1}, \quad w_A(r,\overline{\lambda})^* = w_A(r,\lambda)^{-1} \tag{4.46}$$

hold. In particular, we have

$$\frac{1}{\det w(r,\overline{\lambda})^*} = \det w(r,\lambda), \quad \frac{1}{\det w_A(r,\overline{\lambda})^*} = \det w_A(r,\lambda). \tag{4.47}$$

Recall also that w and w_A are 2×2 matrices, and so (4.46) and (4.47) yield

$$w(r,\lambda) = \big(w(r,\overline{\lambda})^*\big)^{-1} = \big(\det w(r,\lambda)\big)jJ\overline{w(r,\overline{\lambda})}Jj \tag{4.48}$$

$$w_A(r,\lambda) = \big(\det w_A(r,\lambda)\big)jJ\overline{w_A(r,\overline{\lambda})}Jj. \tag{4.49}$$

Here we put $n = 1$ in the definition of J, that is,

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.50}$$

From (1.3) and (1.6) it follows that

$$\det w(r,\lambda) = \exp\left(i \int_0^r \mathrm{Tr}\Big(\sum_{k=1}^m b_k(\lambda - d_k)^{-1} \beta_k(x)^* \beta_k(x)\Big) dx \right)$$
$$= \exp\Big(ir \sum_{k=1}^m b_k(\lambda - d_k)^{-1}\Big), \tag{4.51}$$

where Tr means trend. In a similar way from Lemma 3.6 and Remark 3.5 we get

$$\det w_A(r,\lambda) = \exp\Big(ir \sum_{k=1}^m b_k(\lambda - d_k)^{-1}\Big). \tag{4.52}$$

Finally, substitute (4.48)-(4.52) into (4.45) to obtain

$$\sup_{\Im\mu < -M/4} \| w_A(r,\overline{\lambda}) w(r,\overline{\lambda})^{-1} \| < \infty. \tag{4.53}$$

Using (1.7) one can see that inequalities (4.45) and (4.53) imply (4.41).

Step 4. Taking into account (1.6), (1.7), (4.46), and Lemma 3.6, it is easy to see that for

$$K(r,\mu) := w_A\big(r, \lambda(\mu)\big) w\big(r, \lambda(\mu)\big)^{-1} \tag{4.54}$$

we have

$$\underline{\lim}_{R\to\infty} \frac{1}{\ln R} \ln \ln \sup_{|\mu|=R} \|K(r,\mu)\| \leq 1. \tag{4.55}$$

In view of (4.41) and (4.55), we can apply to $K(r,\mu)$ the Phragmen-Lindelöf theorem. Thus, we see that $\|K(r,\mu)\|$ is uniformly bounded for sufficiently large $|\zeta|$ on the strip $|\eta| \leq M/4$. Using this and inequality (4.41), we have

$$\sup_{|\mu| > M_1} \|K(r,\mu)\| < \infty \tag{4.56}$$

for some $M_1 > 0$.

Let us switch to the variable λ. From (1.7), (4.54), and (4.56) we derive that

$$\widetilde{K}(r,\lambda) := w_A(r,\lambda)w(r,\lambda)^{-1}$$

is bounded in the deleted neighborhood of $\lambda = d_k$. That is, by Theorem 11.4 from [40] the matrix function $\widetilde{K}(r,\lambda)$ is holomorphic at $\lambda = d_k$. Since k $(1 \leq k \leq m)$ is arbitrary, $\widetilde{K}(r,\lambda)$ is an entire function with respect to the variable λ. Moreover, it easy to see that

$$\lim_{\lambda\to\infty} w_A(r,\lambda) = \lim_{\lambda\to\infty} w(r,\lambda) = \lim_{\lambda\to\infty} w_A(r,\lambda)w(r,\lambda)^{-1} = I_2. \tag{4.57}$$

By Liouville's theorem, it follows from (4.57) that the entire function $w_A w^{-1}$ is constant, that is,

$$w_A(r,\lambda) \equiv w(r,\lambda) \quad (0 \leq r \leq l). \tag{4.58}$$

Identity (4.58) implies (4.22). □

The construction of the solution of the inverse problem, which is described in Definition 2.5, is given below.

Theorem 4.3 *Let a vector function $\varphi(\mu)$ be holomorphic in the half-plane $\Im\mu < -\frac{M}{4}$ and admit there an asymptotic representation*

$$\varphi(\mu) = \alpha_0 + \frac{\alpha_1}{\mu} + O\Big(\frac{1}{\mu^2}\Big), \quad \Im\mu < -\frac{M}{4}, \quad \mu\to\infty. \tag{4.59}$$

Then φ is a Weyl function of the unique system (1.2) satisfying conditions (1.4) and (2.1).

To recover (1.2) on an arbitrary fixed interval $[0, l]$ one can use the same procedure as in Theorem 4.2. First, introduce the column vector function $\Phi_2(x)$ $(0 < x < \infty)$ by the Fourier transform (4.3) or, equivalently, (4.4). Here the right-hand side of (4.4) equals 0 for $x < 0$.

Next, substitute $\Phi_2(x)$ into formulas (3.7)-(3.9) to introduce operator $S \in \{L^2_m(0,l),\, L^2_m(0,l)\}$ of the form (3.6). Apply formulas (3.20)-(3.22) to recover operator V from the operators $S(r) = P_r S P_r^*$ $(r \leq l)$.

Then, the matrix functions $\beta_k(x)$ and therefore, system (1.2) is recovered using the formula (3.23), that is, $\beta(x) = V[\Phi_1(x) \quad \Phi_2(x)]$, where Φ_1 is given by the first relation in (3.4).

Proof

Step 1. Let us show that the potential $\beta = V[\Phi_1 \quad \Phi_2]$ is differentiable and relations (1.4) and (2.1) hold.

According to (4.9) and to the asymptotic relation (4.59), the vector function Φ_2 is well-defined by (4.3) and does not depend on the choice of $\eta < -\frac{M}{2}$. So, we can fix some $\eta < -\frac{M}{2}$. Moreover, one can easily see that $\Phi_2(x)$ is twice differentiable and $e^{\eta x}\Phi_2''(x) \in L_m^2(0,\infty)$. Therefore, the matrix function $s(x,u)$ given by (3.8) and (3.9) is continuous. So, $T_r(x,u)$ given by (3.22) is continuous with respect to r, x, and u [19].

The product of the right-hand sides of the first equality in (3.6) (for $l = r$) and of the formula (3.22) equals I_m, which can be written down as the equality

$$s(x,u)B\widetilde{D}^{-1} + B\widetilde{D}T_r(x,u) + \int_0^r s(x,t)T_r(t,u)dt = 0 \quad (x,u \le r) \tag{4.60}$$

for the kernels of the integral operators. In view of the continuity of s and T_r, equality (4.60) is true pointwise. Changing the order of multiplication of the right-hand sides of (3.6) (for $l = r$) and (3.22) we get also

$$B\widetilde{D}^{-1}s(x,u) + T_r(x,u)B\widetilde{D} + \int_0^r T_r(x,t)s(t,u)dt = 0 \quad (x,u \le r). \tag{4.61}$$

From (3.20), (3.21), and (3.23) it follows that

$$\beta(r) = \widetilde{D}^{-\frac{1}{2}}[\Phi_1(r) \quad \Phi_2(r)] + B\widetilde{D}^{\frac{1}{2}}\int_0^r T_r(r,u)[\Phi_1(u) \quad \Phi_2(u)]du. \tag{4.62}$$

Using (4.60)-(4.62), one can show that

$$\beta'(r) = \widetilde{D}^{-\frac{1}{2}}[0 \quad \Phi_2'(r)] + B\widetilde{D}^{\frac{1}{2}}\big(Y_1 + Y_2 + Y_3\big), \tag{4.63}$$

where

$$Y_1 = T_r(r,r)\Phi(r), \quad \Phi(r) = [\Phi_1(r) \quad \Phi_2(r)], \tag{4.64}$$

$$Y_2 = -T_r(r,r)\int_0^r \Big(S(r)^{-1}s(x,r)\Big)^*\Phi(x)dx, \tag{4.65}$$

$$Y_3 = -B\widetilde{D}^{-1}\int_0^r \Big(S(r)^{-1}\frac{\partial}{\partial u}s(x,u)\big|_{u=r}\Big)^*\Phi(x)dx, \tag{4.66}$$

and $S(r)^{-1}$ is applied to the matrix functions columnwise.

Indeed, let us prove that

$$\frac{d}{dr}\int_0^r T_r(r,u)\Phi(u)du = Y_1 + Y_2 + Y_3. \tag{4.67}$$

It is immediate that for $\delta > 0$ we have

$$\begin{aligned}
&\int_0^{r+\delta} T_{r+\delta}(r+\delta,u)\Phi(u)du - \int_0^r T_r(r,u)\Phi(u)du \\
&= \int_r^{r+\delta} T_{r+\delta}(r+\delta,u)\Phi(u)du + \int_0^r \Big(T_{r+\delta}(r+\delta,u) - T_{r+\delta}(r,u)\Big)\Phi(u)du \\
&+ \int_0^r \Big(T_{r+\delta}(r,u) - T_r(r,u)\Big)\Phi(u)du.
\end{aligned} \tag{4.68}$$

As $T_r(x,u)$ is continuous, we get

$$\lim_{\delta\to 0}\delta^{-1}\int_r^{r+\delta} T_{r+\delta}(r+\delta,u)\Phi(u)du = Y_1. \tag{4.69}$$

Substitute $r+\delta$ instead of r into (4.60) to obtain the equality

$$s(x,u)B\widetilde{D}^{-1} + B\widetilde{D}T_{r+\delta}(x,u) + \int_0^{r+\delta} s(x,t)T_{r+\delta}(t,u)dt = 0,$$

which can be rewritten as

$$S(r)T_{r+\delta}(x,u) = -s(x,u)B\widetilde{D}^{-1} - \int_r^{r+\delta} s(x,t)T_{r+\delta}(t,u)dt \tag{4.70}$$

for all fixed values of u ($u \leq r$). According to (4.70) we have

$$T_{r+\delta}(x,r+\delta) - T_{r+\delta}(x,r) = -S(r)^{-1}\Big(\big(s(x,r+\delta) - s(x,r)\big)B\widetilde{D}^{-1}$$

$$+ \int_r^{r+\delta} s(x,t)\Big(T_{r+\delta}(t,r+\delta) - T_{r+\delta}(t,r)\Big)dt\Big). \tag{4.71}$$

Recall that Φ is twice differentiable. Hence, for $u \geq r$ and $x \leq r$ the matrix function $s(x,u)$ is differentiable with respect to u and $\frac{\partial}{\partial u}s(x,u)$ is continuous with respect to x and u. Therefore, formula (4.71) implies

$$\lim_{\delta \to 0} \|\delta^{-1}\Big(T_{r+\delta}(x,r+\delta) - T_{r+\delta}(x,r)\Big) + S(r)^{-1}\frac{\partial}{\partial u}s(x,u)\big|_{u=r}B\widetilde{D}^{-1}\|_{L^2} = 0. \tag{4.72}$$

Here $\|X(x)\|_{L^2}$ denotes the maximum of the norms in $L^2_m(0,r)$ of the columns of the matrix function X. Take into account that $T = T^*$, and thus $T_r(x,u) = T_r(u,x)^*$. So, it follows from (4.66) and (4.72) that

$$\lim_{\delta \to 0} \delta^{-1} \int_0^r \Big(T_{r+\delta}(r+\delta,x) - T_{r+\delta}(r,x)\Big)\Phi(x)dx = Y_3. \tag{4.73}$$

In a similar to (4.71) way, by (4.60) and (4.70) we have

$$T_{r+\delta}(x,r) - T_r(x,r) = -S(r)^{-1}\int_r^{r+\delta} s(x,t)T_{r+\delta}(t,r)dt. \tag{4.74}$$

From (4.74) it follows that

$$\lim_{\delta \to 0} \delta^{-1} \int_0^r \Big(T_{r+\delta}(r,x) - T_r(r,x)\Big)\Phi(x)dx = Y_2. \tag{4.75}$$

According to (4.68), (4.69), (4.73), and (4.75) we get

$$\lim_{\delta \to 0} \delta^{-1}\Big(\int_0^{r+\delta} T_{r+\delta}(r+\delta,u)\Phi(u)du - \int_0^r T_r(r,u)\Phi(u)du\Big) = Y_1 + Y_2 + Y_3. \tag{4.76}$$

Using (4.61), in a similar way one obtains

$$\lim_{\delta \to 0} \delta^{-1}\Big(\int_0^r T_r(r,u)\Phi(u)du - \int_0^{r-\delta} T_{r-\delta}(r-\delta,u)\Phi(u)du\Big) = Y_1 + Y_2 + Y_3. \tag{4.77}$$

From (4.76) and (4.77), the equality (4.67) is immediate. Finally, by (4.62) and (4.67) we have (4.63).

In view of (4.63)-(4.66), β is differentiable and the first relations in (1.4) are valid. According to Remark 3.5 the second relations in (1.4) are fulfilled too. By (4.62) we have $\beta(0) = \widetilde{D}^{-\frac{1}{2}}\Phi(0)$ and so, taking into account (3.4), we get $\beta_{k1}(0) = \big(1 + |\Phi_{k2}(0)|^2\big)^{-\frac{1}{2}} \neq 0$, that is, the inequalities (2.1) are true.

Step 2. Now, let us show that the matrix function β, which is constructed using (3.23), provides the solution of the inverse problem. In view of Definition 1.1, Lemma 3.6, and Remark 3.7 it will suffice to show that for $r \leq l$ and $1 \leq k \leq m$ we have

$$\sup_{\Im\mu < -M/4} \left\| w_A(r,\lambda) \begin{bmatrix} \varphi_k(\mu) \\ 1 \end{bmatrix} \right\| < \infty, \quad \lambda = d_k + \frac{b_k}{2\mu}. \tag{4.78}$$

Taking into account (4.59) one can see that the vector function $\mu^{-1}\varphi\left(\frac{\mu}{2}\right)$ is holomorphic in the half-plane $\eta = \Im\mu < -\frac{M}{2}$ and belongs to $L^2_m(-\infty, \infty)$ with respect to $\zeta = \Re\mu$ for any fixed $\eta < -\frac{M}{2}$. Therefore, similar to the proof of Theorem 4.2 we apply Theorem V from [27] and derive:

$$\mu^{-1}\varphi\left(\frac{\mu}{2}\right) = \mathrm{l.i.m.}_{a\to\infty} \int_0^a e^{-i\mu x} e^{\frac{1}{2}xM} f(x)dx, \quad f \in L^2_m(0,\infty). \tag{4.79}$$

Using Plansherel's theorem and formula (4.3), we express $f(x)$ in (4.79) via $\Phi_2(x)$ and obtain:

$$\frac{i}{\mu}\varphi\left(\frac{\mu}{2}\right) = \int_0^\infty e^{-i\mu x}\Phi_2(x)dx, \quad e^{-\frac{1}{2}xM}\Phi_2(x) \in L^2_m(0,\infty). \tag{4.80}$$

It follows also that the right-hand side of (4.4) equals 0 for $x < 0$, as stated in the theorem. The equality in (4.80) is true pointwise. Recall that according to (4.59) and (4.3), Φ_2 is differentiable and $e^{\eta u}\Phi_2'(u) \in L^2_m(0,\infty)$ for $\eta < -\frac{M}{2}$. Hence, we rewrite (4.80) as:

$$\varphi(\mu) = -\Phi_2(0) - \int_0^\infty e^{-2i\mu u}\Phi_2'(u)du. \tag{4.81}$$

Without loss of generality we shall choose M for the half-plane $\eta < -M/4$, where $\varphi(\mu)$ is treated, so that

$$\Big(\exp\big(\varepsilon - \frac{M}{2}\big)u\Big)\Phi_2'(u) \in L^2_m(0,\,\infty), \quad M > 4\max_{j\neq p}|d_p - d_j|^{-1}, \tag{4.82}$$

$$\sup_{\eta\leq -M/4}\|\varphi(\mu)\| < \infty. \tag{4.83}$$

In view of (3.10), (4.31), and (4.81) we have

$$\big(A_k(r) - \lambda I\big)^{-1}\Pi_k\begin{bmatrix}\varphi_k(\mu)\\ 1\end{bmatrix} = 2b_k\mu e^{2i\mu x}\int_x^\infty e^{-2i\mu u}\Phi_{k2}'(u)du. \tag{4.84}$$

By (4.84) and the first relation in (4.82) we get

$$\left\|\big(A_k(r) - \lambda I\big)^{-1}\Pi_k\begin{bmatrix}\varphi_k(\mu)\\ 1\end{bmatrix}\right\| \leq C(r)|\mu|/\sqrt{|\eta|}. \tag{4.85}$$

By the second relation in (4.82) the inequality $|\lambda - d_p| > |d_k - d_p|/2$ is true for $p \neq k$, $\lambda = d_k + \frac{b_k}{2\mu}$, and $\eta < -M/4$, that is, the resolvents $\big(A_p(r) - \lambda I\big)^{-1}$ $(p \neq k)$ are bounded. Therefore, the inequalities (4.83) and (4.85) imply

$$\left\|\big(A(r) - \lambda I\big)^{-1}\Pi\begin{bmatrix}\varphi_k(\mu)\\ 1\end{bmatrix}\right\| \leq C_1(r)|\mu|/\sqrt{|\eta|}. \tag{4.86}$$

As $|\lambda - \overline{\lambda}| = -\eta|\mu|^{-2}$, from (4.30), (4.83), and (4.86) the inequality

$$\sup[\overline{\varphi_k(\mu)} \quad 1]w_A(r,\lambda)^* w_A(r,\lambda)\begin{bmatrix}\varphi_k(\mu)\\ 1\end{bmatrix} < \infty$$

is immediate. Thus, (4.78) is valid, and the solution of the inverse problem can be obtained via (3.23).

The uniqueness of the solution of the inverse problem is stated in Theorem 2.6. □

In a way similar to [34,35], the procedure to solve the inverse problem grants also a Borg-Marchenko-type theorem.

Theorem 4.4 *Let the $\mathbb{C}^m$-valued vector functions $\varphi(\mu,1)$ and $\varphi(\mu,2)$ be holomorphic in the half-plane $\Im\mu < -M/4$ $(M>0)$ and satisfy (4.59). Suppose that on some ray $c\Im\mu = \Re\mu$ $(c=\overline{c},\ \Im\mu < -M/4)$ we have*

$$\|\varphi(\mu,1)-\varphi(\mu,2)\| = e^{-2i\mu l}O(1) \quad \text{for}\, |\mu| \to \infty. \tag{4.87}$$

Then $\varphi(\mu,1)$ and $\varphi(\mu,2)$ are Weyl functions of systems (1.2) with potentials $\beta(x,1)$ and $\beta(x,2)$, respectively, which satisfy (1.4), (2.1) and the additional equality

$$\beta(x,1) \equiv \beta(x,2) \quad (0<x<l). \tag{4.88}$$

Proof

According to Theorem 4.3 the functions $\varphi(\mu,1)$ and $\varphi(\mu,2)$ are Weyl functions of systems (1.2) with potentials $\beta(x,1)$ and $\beta(x,2)$, which satisfy (1.4) and (2.1). Denote by $\Phi_2(x,j)$ the matrix function generated via (4.3) by $\varphi(\mu,j)$ $(j=1,2)$, and introduce the matrix function

$$\nu(\mu) := \int_0^l \big(\exp 2i\mu(l-u)\big)\Big(\Phi_2(u,1)-\Phi_2(u,2)\Big)du. \tag{4.89}$$

Next, we shall show that $\nu(\mu)=0$. Indeed, in view of (4.81) and (4.87) it is immediate that $\Phi_2(0,1)=\Phi_2(0,2)$. Hence, from (4.81) and (4.89) we derive

$$\nu(\mu) = e^{2i\mu l}\big(\varphi(\mu,2)-\varphi(\mu,1)\big) + \int_l^\infty \big(\exp 2i\mu(l-u)\big)\Big(\Phi_2(u,1)-\Phi_2(u,2)\Big)du. \tag{4.90}$$

It is immediate from (4.89) that $\|\nu(\mu)\|$ is bounded on the line $\Im\mu = -M/4$. Using (4.87) and (4.90) we see that $\|\nu(\mu)\|$ is bounded on the ray $c\Im\mu = \Re\mu$ $(\Im\mu < -M/4)$. Thus, by the Phragmen-Lindelöf theorem $\|\nu(\mu)\|$ is bounded in the half-plane $\Im\mu \le -M/4$. Moreover, by (4.89) $\|\nu(\mu)\|$ is bounded for $\Im\mu > -M/4$, that is, $\nu(\mu)$ is an entire function bounded on $\mathbb{C}$. Therefore, $\nu(\mu)$ is a constant. As $\lim_{\eta\to\infty}\nu(\mu)=0$, so we obtain $\nu(\mu)\equiv 0$, that is

$$\Phi_2(x,1) \equiv \Phi_2(x,2) \quad (0<x<l). \tag{4.91}$$

By the procedure to solve the inverse problem (see Theorem 4.3) formula (4.91) implies (4.88). $\square$

5 Weyl set

Condition (2.1) is not necessary in the considerations of Sections 2, 3 and 4. Taking into account the identity in (1.3), one can choose two sets of natural numbers N_1 and N_2, so that

$$\beta_{k1}(0) \neq 0 \quad \text{for} \quad k \in N_1, \quad \beta_{k2}(0) \neq 0 \quad \text{for} \quad k \in N_2, \tag{5.1}$$

$$N_1 \cap N_2 = \emptyset, \quad N_1 \cup N_2 = \{1, 2, \ldots, m\}. \tag{5.2}$$

The procedure to solve the inverse problem is easily modified for that case. Such a modification is discussed below. We also introduce a notion of the Weyl set, and the system (1.2), which satisfies (1.3), is recovered from its Weyl set.

Definition 5.1 *The set $\varkappa$ of the pairs*

$$\varkappa = \{\beta_k(0), \widetilde{\psi}_k(\mu) \,|\, 1 \leq k \leq m\}, \tag{5.3}$$

where $\widetilde{\psi}_k$ is given by (2.18), is called a Weyl set of the system (1.2), which satisfies (1.3).

If the functions $\widetilde{\psi}_k$ in the Weyl sets $\varkappa$ and $\breve{\varkappa}$ coincide and the vectors $\beta_k(0)$ and $\breve{\beta}_k(0)$ in $\varkappa$ and $\breve{\varkappa}$, respectively, satisfy the equalities $\breve{\beta}_k(0) = c_k \beta_k(0)$, $|c_k| = 1$ $(1 \leq k \leq m)$, we say that $\varkappa = \breve{\varkappa}$, as $\varkappa$ and $\breve{\varkappa}$ correspond to the same system. After we exclude this arbitrariness, the Weyl set is uniquely defined by system (1.2), (1.3). (See the construction of $\widetilde{\psi}_k$ in Section 2 and the proof of Theorem 2.2 in [26].)

The next lemma is proved in a quite similar way to Lemma 3.6.

Lemma 5.2 *Let the S-node be given by the formulas (3.1)-(3.3) and (3.5), where*

$$\Phi_{k1}(x) \equiv 1, \quad \Phi'_{k2}(x) \in L^2(0, l) \quad \text{for} \quad k \in N_1, \tag{5.4}$$

$$\Phi_{k2}(x) \equiv 1, \quad \Phi_{k1}'(x) \in L^2(0,l) \quad \text{for} \quad k \in N_2. \tag{5.5}$$

Define the reductions $A(r)$, $S(r)$ and $\Pi(r)$ $(r < l)$ of the operators from this S-node in the same way as it was done after Remark 3.2 in Section 3.

Then w_A of the form (3.29) satisfies (3.30), where β is given by (3.23) and the operator V in (3.23) is obtained via (3.19)-(3.22).

Remark 5.3 *The operator S is uniquely recovered from the operator identity (3.5), the corresponding formulas from Proposition 3.1 are easily modified for the more general case (5.4), (5.5).*

In view of Lemma 5.2, the proof of the next theorem is similar to the proof of Theorem 4.2.

Theorem 5.4 *Let system (1.2) satisfy conditions (1.3) and let N_1 and N_2 be chosen so that (5.1) and (5.2) hold.*

Then, system (1.2) is uniquely recovered from its Weyl set. To recover (1.2) on an arbitrary fixed interval $[0,\, l]$ we use the following procedure.

First, the entries of the column vector functions $\Phi_1(x)$ and $\Phi_2(x)$ $(0 < x < \infty)$ are introduced by the equalities:

$$\Phi_{k1}(x) \equiv 1, \quad \Phi_{k2}(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \mu^{-1} e^{i\mu x} \varphi_k\left(\frac{\mu}{2}\right) d\zeta \quad \text{for}\, k \in N_1; \tag{5.6}$$

$$\Phi_{k1}(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \mu^{-1} e^{i\mu x} \phi_k\left(\frac{\mu}{2}\right) d\zeta, \quad \Phi_{k2}(x) \equiv 1 \quad \text{for}\, k \in N_2; \tag{5.7}$$

$$\varphi_k(\mu) = \frac{\overline{\beta_{k1}(0)}\widetilde{\psi}_k(\mu) - \beta_{k2}(0)}{\overline{\beta_{k2}(0)}\widetilde{\psi}_k(\mu) + \beta_{k1}(0)}, \quad \phi_k(\mu) = \frac{\overline{\beta_{k2}(0)}\widetilde{\psi}_k(\mu) + \beta_{k1}(0)}{\overline{\beta_{k1}(0)}\widetilde{\psi}_k(\mu) - \beta_{k2}(0)}, \tag{5.8}$$

where $\mu = \zeta + i\eta$ and $-\eta > 0$ is sufficiently large.

Next, we introduce the S-node by formulas (3.1)-(3.3) and (3.5). The operator S has the form (3.6), where the definition (3.7) of $\widetilde{D}$ is modified:

$$\widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2, \quad \widetilde{D}_p = \mathrm{diag}\{|\Phi_{1p}(0)|^2, |\Phi_{2p}(0)|^2, \dots, |\Phi_{mp}(0)|^2\} \quad (p = 1, 2). \tag{5.9}$$

Finally, we put

$$\breve{\beta}(x) = \begin{bmatrix} \breve{\beta}_1(x) \\ \cdots \\ \breve{\beta}_m(x) \end{bmatrix} = (V\Phi)(x) \quad (0 \leq x \leq l), \tag{5.10}$$

where the operator V is obtained via (3.19)-(3.22).

The equalities

$$\breve{\beta}_k(x)^* \breve{\beta}_k(x) \equiv \beta_k(x)^* \beta_k(x) \quad (1 \leq k \leq m). \tag{5.11}$$

are valid, that is, system (1.2) is recovered by the procedure, which is described above.

6 Sine-Gordon equation

The initial value problem for the sine-Gordon equation in the light cone coordinates $\omega_{xt} = \sin\, \omega$ ($\omega_x := \frac{\partial}{\partial x}\omega$) was treated in [1]. The initial value problem (with initial conditions tending to zero) for the sine-Gordon equation in laboratory coordinates

$$\omega_{xx} - \omega_{tt} = \sin \omega \tag{6.1}$$

was investigated by Faddeev, Takhtajan and Zakharov (see [44] and further references in [13]). Notice also that the Goursat problem for the equation $\omega_{xt} = \sin\, \omega$, which is treated on the characteristics $t = 0$ and $x = -\infty$ in [20], is equivalent to the Cauchy problem for equation (6.1). In this section we

consider (6.1) under boundary conditions $\omega(0,t)=\omega_0(t)$ and $\omega_x(0,t)=\omega_1(t)$. (The boundary value problem is clearly equivalent to the initial value problem after the change of variables and the change $\omega \to \omega+\pi$.) We do not require that ω tends to zero and only the boundedness of ω_x and ω_t is needed.

Equation (6.1) admits a zero curvature representation

$$G_t(x,t,\lambda)-F_x(x,t,\lambda)+G(x,t,\lambda)F(x,t,\lambda)-F(x,t,\lambda)G(x,t,\lambda)=0. \tag{6.2}$$

We can modify the auxiliary systems

$$w_x(x,t,\lambda)=G(x,t,\lambda)w(x,t,\lambda), \quad w_t(x,t,\lambda)=F(x,t,\lambda)w(x,t,\lambda), \tag{6.3}$$

so that they have the form (1.2). Namely, put

$$\begin{aligned} &G(x,t,\lambda)=i\sum_{k=1}^{2} b_k(\lambda-d_k)^{-1}\Big(\beta_k(x,t)^*\beta_k(x,t)\Big), \\ &d_1=-d_2=1, \quad b_1=b_2=1, \end{aligned} \tag{6.4}$$

$$\begin{aligned} &F(x,t,\lambda)=i\sum_{k=1}^{2} b_k(\lambda-d_k)^{-1}\Big(\beta_k(x,t)^*\beta_k(x,t)\Big), \\ &d_1=-d_2=1, \quad b_1=-b_2=1, \end{aligned} \tag{6.5}$$

where

$$\beta_1(x,t)=\frac{1}{\sqrt{2}}[1 \quad i\,e^{i\omega(x,t)/2}]q(x,t), \qquad \beta_2(x,t)=\frac{1}{\sqrt{2}}[1 \quad ie^{-i\,\omega(x,t)/2}]q(x,t), \tag{6.6}$$

the 2×2 matrix function q satisfies the equations

$$q_x(x,t)=\breve{G}(x,t)q(x,t), \quad q_t(x,t)=\breve{F}(x,t)q(x,t), \quad q(0,0)=I_2, \tag{6.7}$$

$$\breve{G}:=-i\left(\frac{\omega_t}{4}j+\frac{1}{2}\sin\left(\frac{\omega}{2}\right)J\right), \quad \breve{F}:=-i\frac{\omega_x}{4}j+\frac{1}{2}\cos\left(\frac{\omega}{2}\right)Jj, \tag{6.8}$$

and J, j are given in (4.50). It is easily checked that the sine-Gordon equation (6.1) is equivalent to the compatibility condition $\breve{G}_t-\breve{F}_x+\breve{G}\breve{F}-\breve{F}\breve{G}=0$ of the equations (6.7). Moreover, direct calculation shows that relations (6.6)

and (6.7) imply (6.2), which is the compatibility condition for (6.3). Thus, if (6.1) holds, equations (6.3) are compatible.

Introduce the 2×2 matrix functions $Z(x,t,\lambda)$,

$$Z(t,\lambda) := Z(0,t,\lambda) = \{Z_{ij}(t,\lambda)\}_{i,j=1}^2, \quad \text{and} \quad Y(x,t,\lambda) = \{Y_{ij}(x,t,\lambda)\}_{i,j=1}^2$$

by the equations

$$\begin{aligned} Y_x(x,t,\lambda) &= G(x,t,\lambda)Y(x,t,\lambda), \quad Y(0,t,\lambda) \equiv I_2; \\ Z_t(x,t,\lambda) &= F(x,t,\lambda)Z(x,t,\lambda), \quad Z(x,0,\lambda) \equiv I_2. \end{aligned} \tag{6.9}$$

The matrix functions $Q_k(x,t)$ $(k=1,2)$ are connected with $\beta_k(x,t)$ by the equalities (2.3). According to (2.3) and (6.6)-(6.8) the boundary conditions

$$\omega(0,t) = \omega_0(t), \quad \omega_x(0,t) = \omega_1(t) \quad (-\infty < t < \infty) \tag{6.10}$$

uniquely define $\breve{F}(0,t)$, $q(0,t)$, $\beta_k(0,t)$ and $Q_k(0,t)$. If we recover also $\widetilde{\psi}_k(t,\mu)$ for each $-\infty < t < \infty$, we have a Weyl set for each t. First, we recover $F(0,t,\lambda)$ and $Z(t,\lambda)$, using formulas (6.5) and (6.9), and put

$$U_k(x,t,\mu) := \exp\{(-1)^k it\mu\}Q_k(x,t)Z(x,t,\lambda)Q(x,0)^*, \quad \mu = \big(2(\lambda - d_k)\big)^{-1}, \tag{6.11}$$

$$U_k(t,\mu) = \{u_{jp}(t,\mu,k)\}_{j,p=1}^2 := U_k(0,t,\mu). \tag{6.12}$$

The matrix functions $U_k(t,\mu)$ are uniquely recovered from (6.10) too.

Theorem 6.1 *Let the function $\omega(x,t)$ have continuous second derivatives in the semi-plane $x \geq 0$ and satisfy the sine-Gordon equation (6.1) and boundary conditions (6.10). Assume also that*

$$\sup_{x \geq 0}(|\omega_x(x,t)| + |\omega_t(x,t))|) < \infty. \tag{6.13}$$

Then, $\cos\omega(x,t)$ $(x \geq 0)$ is uniquely recovered from (6.10). For this purpose construct $U_k(t,\mu)$ $(k=1,2)$ using (6.5)-(6.12). There is $M_1 > 0$, such that for $-\Im\mu > M_1$ we have

$$\widetilde{\psi}_1(0,\mu) = -\lim_{t \to \infty} \frac{u_{12}(t,\mu,1)}{u_{11}(t,\mu,1)}, \quad \widetilde{\psi}_2(0,\mu) = -\lim_{t \to -\infty} \frac{u_{12}(t,\mu,2)}{u_{11}(t,\mu,2)}. \tag{6.14}$$

The functions $\widetilde{\psi}_k(t,\mu)$ are given by the formulas

$$\widetilde{\psi}_k(t,\mu) = \frac{u_{11}(t,\mu,k)\widetilde{\psi}_k(0,\mu) + u_{12}(t,\mu,k)}{u_{21}(t,\mu,k)\widetilde{\psi}_k(0,\mu) + u_{22}(t,\mu,k)}. \tag{6.15}$$

By formulas (2.3), (6.6)-(6.8), (6.14) and (6.15) we recover the Weyl set for each t. Finally, we recover the functions $\beta_k(x,t)$ (up to factors $c_k(x,t)$ such that $|c_k| = 1$, $c_k(0,t) = 1$) using Theorem 5.4. It follows that

$$\cos\omega(x,t) = 2\beta_1(x,t)\beta_2(x,t)^*\beta_2(x,t)\beta_1(x,t)^* - 1. \tag{6.16}$$

Proof

Step 1. In this step we shall prove (6.15). Note that as ω has continuous second derivatives, so according to (6.4)-(6.8) the matrix functions G and F are continuously differentiable. Therefore, the formula (1.6) on p. 168 in [39] implies:

$$Y(x,t,\lambda) = Z(x,t,\lambda)Y(x,0,\lambda)Z(t,\lambda)^{-1}. \tag{6.17}$$

By (1.6), (2.4), and (6.9) we have $w(x,t,\lambda) = Y(x,t,\lambda)$,

$$Q(x,t)^* = Q(x,t)^{-1}, \quad Y(x,t,\overline{\lambda})^* = Y(x,t,\lambda)^{-1}, \quad Z(x,t,\overline{\lambda})^* = Z(x,t,\lambda)^{-1}. \tag{6.18}$$

Hence, taking into account (2.2) and (2.12) we have

$$\mathfrak{A}(r,t,\mu) = e^{ir\mu}Q(0,t)Y(r,t,\overline{\lambda})^*Q(r,t)^*. \tag{6.19}$$

From (6.11), (6.12), (6.17), and (6.19) it follows that

$$\mathfrak{A}_k(r,t,\mu) = U_k(t,\mu)\mathfrak{A}_k(r,0,\mu)U_k(r,t,\mu)^{-1}. \tag{6.20}$$

In view of (6.9) and (6.11) it is easy to see that

$$\frac{\partial}{\partial t}U_k(r,t,\mu) = \Big((-1)^{k+1}i\mu j + \Big(\frac{\partial}{\partial t}Q_k(r,t)\Big)Q_k(r,t)^* \tag{6.21}$$
$$+(-1)^k iQ_k(r,t)\frac{\beta_p(r,t)^*\beta_p(r,t)}{\lambda - d_p}Q_k(r,t)^*\Big)U_k(r,t,\mu),$$

where k and p take values 1 and 2, $p \neq k$. According to (6.7) and (6.8) the equalities $\frac{\partial}{\partial t}(q^*q) = 0$ and $\frac{\partial}{\partial x}(q^*q) = 0$ are true and $q(0,0) = I_2$. Therefore, it is immediate that q is unitary:

$$q(x,t)^*q(x,t) \equiv I_2. \tag{6.22}$$

By (6.6)-(6.8), (6.13) and (6.22) we have

$$\sup \| \frac{\partial}{\partial t}\beta_k(x,t)\| < \infty \quad (x \geq 0, \, -\infty < t < \infty), \quad k = 1, \, 2. \tag{6.23}$$

Taking into account (6.21) and (6.23), in a way similar to the proof of (2.11) we derive

$$(-1)^{k+1}\frac{\partial}{\partial t}\Big(U_k(r,t,\mu)^*jU_k(r,t,\mu)\Big) \geq \big(i(\mu - \overline{\mu}) - \widehat{M}\big)U_k(r,t,\mu)^*U_k(r,t,\mu) > 0 \tag{6.24}$$

for some $\widehat{M} > 0$ and $\Im\mu < -\widehat{M}/2$. From (6.24) it follows that

$$U_1(r,t,\mu)^*jU_1(r,t,\mu) > j \quad \text{for}\, t > 0; \quad U_2(r,t,\mu)^*jU_2(r,t,\mu) > j \quad \text{for}\, t < 0. \tag{6.25}$$

From the first inequality in (6.25) we have $\big(U_1(r,t,\mu)^*\big)^{-1}jU_1(r,t,\mu)^{-1} < j$,

$$[\theta(\mu)^* \quad 1]\big(U_1(r,t,\mu)^*\big)^{-1}jU_1(r,t,\mu)^{-1}\begin{bmatrix} \theta(\mu) \\ 1 \end{bmatrix} < 0 \quad \text{for } t > 0, \, |\theta| \leq 1. \tag{6.26}$$

In a similar way, from the second inequality in (6.25) we derive

$$[\theta(\mu)^* \quad 1]\big(U_2(r,t,\mu)^*\big)^{-1}jU_2(r,t,\mu)^{-1}\begin{bmatrix} \theta(\mu) \\ 1 \end{bmatrix} < 0 \quad \text{for } t < 0, \, |\theta| \leq 1. \tag{6.27}$$

By (6.26) and (6.27) we obtain the inequalities

$$|\chi_1(r,t,\mu)| < 1 \quad \text{for} \quad t > 0; \quad |\chi_2(r,t,\mu)| < 1 \quad \text{for} \quad t < 0 \quad (\Im\mu < -\frac{\widehat{M}}{2}) \tag{6.28}$$

for the functions

$$\chi_k(r,t,\mu) := \frac{(U_k^{-1})_{11}(r,t,\mu)\theta(\mu) + (U_k^{-1})_{12}(r,t,\mu)}{(U_k^{-1})_{21}(r,t,\mu)\theta(\mu) + (U_k^{-1})_{22}(r,t,\mu)}, \quad k = 1, 2; \; |\theta| \leq 1.$$

In view of (2.13) and (2.18) we have

$$\widetilde{\psi}_k(t,\mu)=\lim_{r\to\infty}\frac{\mathfrak{A}_{11}(r,t,\mu)\theta(\mu)+\mathfrak{A}_{12}(r,t,\mu)}{\mathfrak{A}_{21}(r,t,\mu)\theta(\mu)+\mathfrak{A}_{22}(r,t,\mu)}, \quad |\theta(\mu)|\leq 1, \quad \Im\mu<-\frac{M}{4}. \tag{6.29}$$

From (6.20) it follows that the linear fractional transformation on the right-hand side of (6.29) can be written down as the superposition of the three linear fractional transformations, the first of which transforms θ into χ_k. Using (6.28) we see that the second transformation transforms χ_k into $\psi_k(r,0,\mu)$ for $k=1$ and $t>0$ as well as for $k=2$ and $t<0$. In the limit, it follows from (6.20), (6.28), and (6.29) that (6.15) is true for $k=1$, $t>0$ and for $k=2$, $t<0$ $\big(-\eta>M_0=\max(\frac{\widehat{M}}{2},\,\frac{M}{4})\big)$.

To prove (6.15) for $k=1$, $t<0$ and for $k=2$, $t>0$ rewrite (6.20) in the form

$$\mathfrak{A}_k(r,t,\mu)U_k(r,t,\mu)=U_k(t,\mu)\mathfrak{A}_k(r,0,\mu), \tag{6.30}$$

and use the inequalities

$$U_1(r,t,\mu)^*jU_1(r,t,\mu)<j \text{ for } t<0; \quad U_2(r,t,\mu)^*jU_2(r,t,\mu)<j \text{ for } t>0, \tag{6.31}$$

which are immediate from (6.24). By (6.31) one can see that

$$|\breve{\chi}_1(r,t,\mu)|<1 \quad \text{for} \quad t<0; \quad |\breve{\chi}_2(r,t,\mu)|<1 \quad \text{for} \quad t>0 \quad (\Im\mu<-\frac{\widehat{M}}{2}), \tag{6.32}$$

where

$$\breve{\chi}_k(r,t,\mu):=\frac{(U_k)_{11}(r,t,\mu)\theta(\mu)+(U_k)_{12}(r,t,\mu)}{(U_k)_{21}(r,t,\mu)\theta(\mu)+(U_k)_{22}(r,t,\mu)}, \quad k=1,2;\ |\theta|\leq 1.$$

Now, consider the linear fractional transformations of θ, where the coefficients are the entries of the left-hand side and right-hand side of (6.30), respectively. These linear fractional transformations coinside, and in the limit (as r tends to infinity) we obtain (6.15) for $k=1$, $t<0$ and for $k=2$, $t>0$. Thus, (6.15) is proved.

Step 2. By (2.14) we have $|\widetilde{\psi}_k(t,\mu)|<1$ $(\Im\mu<-\frac{M}{4})$. Hence, in view of

(6.15) we obtain

$$[\widetilde{\psi}_k(0,\mu)^* \quad 1]U_k(t,\mu)^* j U_k(t,\mu) \begin{bmatrix} \widetilde{\psi}_k(0,\mu) \\ 1 \end{bmatrix} \leq 0. \tag{6.33}$$

Recall that (6.15) holds for $-\Im\mu > M_0 \geq \frac{\widehat{M}}{2}$. From (6.24) it follows that for $-\Im\mu > M_1 = \frac{\varepsilon}{2} + M_0$ $(\varepsilon > 0)$ the inequalities

$$U_1(t,\mu)^* j U_1(t,\mu) - j \geq \varepsilon \int_0^t U_1(s,\mu)^* U_1(s,\mu) ds \quad (t > 0), \tag{6.34}$$

$$U_2(t,\mu)^* j U_2(t,\mu) - j \geq \varepsilon \int_t^0 U_2(s,\mu)^* U_2(s,\mu) ds \quad (t < 0) \tag{6.35}$$

are true. According to (6.33)–(6.35), for any μ such that $-\Im\mu > M_1$ we have

$$\int_0^\infty [\widetilde{\psi}_1(0,\mu)^* \quad 1]U_1(s,\mu)^* U_1(s,\mu) \begin{bmatrix} \widetilde{\psi}_1(0,\mu) \\ 1 \end{bmatrix} ds < \infty, \tag{6.36}$$

$$\int_{-\infty}^0 [\widetilde{\psi}_2(0,\mu)^* \quad 1]U_2(s,\mu)^* U_2(s,\mu) \begin{bmatrix} \widetilde{\psi}_2(0,\mu) \\ 1 \end{bmatrix} ds < \infty. \tag{6.37}$$

By (6.13), (6.21), (6.36), and (6.37) the inequalities

$$\sup_{t>0} \left\| U_1(t,\mu) \begin{bmatrix} \widetilde{\psi}_1(0,\mu) \\ 1 \end{bmatrix} \right\| < \infty, \quad \sup_{t<0} \left\| U_2(t,\mu) \begin{bmatrix} \widetilde{\psi}_2(0,\mu) \\ 1 \end{bmatrix} \right\| < \infty \tag{6.38}$$

are valid. Inequalities (6.34) and (6.35) imply that

$$|u_{11}(t,\mu,1)|^2 > 1 + \varepsilon t \quad (t > 0), \quad |u_{11}(t,\mu,2)|^2 > 1 - \varepsilon t \quad (t < 0). \tag{6.39}$$

From (6.38) and (6.39) follows (6.14).

In view of (6.6) and (6.22) we obtain

$$2\beta_1\beta_2^* = 1 + \cos\omega + i\sin\omega, \quad \text{i.e.,} \quad 2|\beta_1\beta_2^*|^2 = 1 + \cos\omega.$$

Hence, the equality (6.16) is immediate. $\square$

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Fakultät für Mathematik, Universität Wien,
Nordbergstrasse 15, A-1090 Wien, Austria.
al_sakhnov@yahoo.com